Journal of Applied Mechanics and Technical Physics, Vol. 38, No. 3, 1997

PROBLEM OF DETERMINING THE ROUGHNESS FACTOR FOR FLOW IN AN OPEN CHANNEL

T. P. Pukhnachova

UDC532.54:517.925

The description of an unsteady flow of a fluid in an open channel is usually based on a one-dimensional system of Saint-Venant equations [1, 2]. The adequacy of this mathematical model depends, to a considerable extent, on the accuracy of assigning the physical parameters that enter this model. In particular, the Chezy coefficient, or the roughness factor expressed in terms of this coefficient, is one of the most important parameters. Precisely this parameter is difficult to measure immediately. The roughness factor is often considered a constant whose average value is found from field observations using the Chezy formula [1, 2]. Another approach was proposed by Voyevodin and Nikiforovskaya in [3] where a series of field observations is used to construct a resolving function. This function was then minimized, and its minimum value was taken as the roughness factor.

In the present paper, the Chezy coefficient is assumed to be a function which depends on the spatial variable. The problem of defining this function is regarded as a coefficient inverse problem for a hyperbolic system of equations. The theory of this type of problems was developed by Romanov [4].

1. Formulation of the Problem. We shall consider a system of Saint-Venant equations under the following assumptions: the flow has a subcritical velocity v and flows in a rectangular channel with constant cross section and zero slope of the bottom. The width of a free surface B is a known constant. This special model allows one to simplify mathematical procedures. The more general case and the necessary changes in the proof are considered in Sec. 4.

Let us denote time by t, the coordinate along the channel by x, the level of the free surface which coincides with the depth at zero slope of bottom by $h(x,t)$, the flow rate by $Q(x,t)$, and the gravity acceleration by g. We assume that a dry channel is absent, i.e., $h > 0$ for all x and t.

The Chezy coefficient $C(x)$ that determines the friction force can have a different form, depending on the hydraulic formula used. For example, the roughness factor $n(x)$ can be related to $C(x)$ by the Manning formula: $C = R^{1/6}/n(x)$ [$R = R(h)$ is the hydraulic radius]. We shall assume that Q is a constant-sign function. Without loss of generality, we can assume that $Q > 0$. The subcritical flow velocity makes it possible to assume the Froude number $\mathrm{Fr} = (v^2/gh_{\mathrm{men}})$ to be much smaller than unity. Therefore, the convective term $\partial(v^2/2g)/\partial x$ which enters the equation of dynamic equilibrium can be disregarded. Under such assumptions, the system of Saint-Venant equations is of the form

$$B\frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} = 0, \qquad \frac{\partial Q}{\partial t} + gBh\frac{\partial h}{\partial x} = -\frac{1}{C^2(x)}\frac{Q^2}{h}\frac{g}{BR}. \tag{1.1}$$

The Chezy coefficient $C(x)$ which enters the right-hand side is to be determined. Let us introduce a function $a(x) \equiv 1/C^2(x)$.

For system (1.1), we shall consider the initial boundary-value problem

$$h(0,t) = f_1(t), \qquad Q(0,t) = f_2(t), \qquad f_1, f_2 \in C^2[0,2T], \qquad f_1, f_2 > 0; \tag{1.2}$$

$$h(x,T) = \varphi_0(x), \qquad \varphi_0 \in C^2[0,L], \qquad \varphi_0 > 0, \qquad 0 \leqslant x \leqslant L, \qquad 0 \leqslant t \leqslant 2T \tag{1.3}$$

Novosibirsk State University, Novosibirsk 630090. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 38, No. 3, pp. 93–98, May–June, 1997. Original article submitted January 24, 1996; revision submitted March 19, 1996.

with any final T and L.

Relations (1.1) and (1.2) represent the standard Cauchy problem in terms of the variable x for a quasi-linear hyperbolic system, and we call it the *direct problem*. Condition (1.3) is required for resolvability of the inverse problem.

For system (1.1), we introduce the corresponding Riemann invariants:

$$r = \frac{2}{3}Bg^{1/2}h^{3/2} + Q, \qquad s = -\frac{2}{3}Bg^{1/2}h^{3/2} + Q.$$

The functions Q and h can be readily expressed in terms of r and s:

$$Q = \frac{r+s}{2}, \qquad h = \left[\frac{3(r-s)}{4Bg^{1/2}}\right]^{2/3}.$$

The direct problem (1.1) and (1.2) for the Riemann invariant is written as the Cauchy problem in terms of x:

$$\frac{\partial r}{\partial x} + u(r-s)\frac{\partial r}{\partial t} = -a(x)v(r,s), \qquad \frac{\partial s}{\partial x} - u(r-s)\frac{\partial s}{\partial t} = a(x)v(r,s); \tag{1.4}$$

$$r(0,t) = \frac{2}{3}Bf_1(t)(gf_1(t))^{1/2} + f_2(t) = r_0(t), \tag{1.5}$$

$$s(0,t) = -\frac{2}{3}Bf_1(t)(gf_1(t))^{1/2} + f_2(t) = s_0(t).$$

Here

$$u(r-s) = (gh(r,s))^{-1/2} = \left(\frac{4Bg}{3(r-s)}\right)^{1/3}; \quad v(r,s) = \frac{Q^2(r,s)}{h(r,s)}\frac{g}{BR} = \frac{(r+s)^2}{(r-s)}\frac{g}{3R}.$$

We shall write, instead of (1.3), an additional condition in the form

$$r(x,T) - s(x,T) = \frac{4}{3}Bg^{1/2}(\varphi_0(x))^{3/2} \equiv \varphi(x). \tag{1.6}$$

It is well known that problem (1.1) and (1.2) in terms of Q and h and problem (1.4) and (1.5) in terms of r and s are equivalent. Note that condition (1.2) ensures the validity of the inequalities $u > 0$ and $v > 0$ at least for small x.

In what follows, the problem of defining the functions $r(x,t)$, $s(x,t)$, and $a(x)$ that satisfy relations (1.4)–(1.6) will be called the *inverse problem*.

A transition from the desired functions to the Riemann invariants can be performed for any quasi-linear hyperbolic system with two independent variables. In studying system (1.1), we take into account the concrete form of the functions $r(x,t)$ and $s(x,t)$ and how the desired coefficient $a(x)$ enters the right-hand side. First, this allows us to propose an additional condition having a reasonable physical sense and, second, to obtain an additional relation to define $a(x)$. Therefore, although the further considerations are applicable to a sufficiently arbitrary quasi-linear system, each particular case needs a separate consideration.

2. Existence and Uniqueness of the Solution. The solution of the inverse problem (1.4)–(1.6) is closely related to the solution of the direct problem (1.4) and (1.5). The direct problem has been studied by many authors. In particular, Rozhdestvenskii and Yanenko [5] elucidated the resolvability conditions and indicated the existence region for the solution. For this purpose, they developed an iteration process and established its convergence and, in addition, they showed that the solution is uniformly limited together with secondary derivatives and found the domain in which the solution is unique for all iterations. Using these results, we obtain the conditions that are necessary to resolve the inverse problem.

Lemma. *For the inverse problem* (1.1)–(1.2) *to be resolved, it is necessary that the initial and boundary data* (1.2), (1.3) *satisfy the conditions*

$$\varphi_0(x) \in C^2[0,L], \qquad \frac{\partial^k \varphi_0(0)}{\partial x^k} = \frac{\partial^k f_1(T)}{\partial t^k} \quad (k = 0,\, 1,\, 2),$$

$$a(0)\frac{g}{BR}\frac{f_1^2(T)}{f_2(T)} + f_2'(T) + gB\varphi_0(0)\varphi_0'(0) = 0.$$

Here and below, the dash refers to a variable. The latter equality can be used to find the quantity $a(0)$.

The characteristics $t_r(\xi, x, t)$ and $t_s(\xi, x, t)$ of system (1.4) are the solutions of the Cauchy problem:

$$\frac{\partial t_r}{\partial \xi} = u(r(\xi, t_r(\xi, x, t)) - s(\xi, t_r(\xi, x, t))), \quad \frac{\partial t_s}{\partial \xi} = -u(r(\xi, t_s(\xi, x, t)) - s(\xi, t_s(\xi, x, t))),$$

$$t_r(x, x, t) = t, \qquad t_s(x, x, t) = t.$$

Let us denote by U a set of functions $f(x,t) \in C^2[0, L] \times [0, 2T]$ such that $\|f\|_{C^2} < M$. For U, we shall find the domain $G(M)$ in which the solution of the inverse problem exists: $G(M) = \{0 \leqslant x \leqslant L,\ T_1(\xi) \leqslant t \leqslant T_2(\xi)\}$, where $T_1(\xi)$ and $T_2(\xi)$ satisfy the following Cauchy problems:

$$\frac{\partial T_1}{\partial \xi} = \max\{u\}, \quad T_1(0) = 0, \qquad \frac{\partial T_2}{\partial \xi} = -\max\{u\}, \quad T_2(0) = 2T.$$

Theorem 1. *Let the functions f_1, f_2, and φ_0 satisfying the conditions of the lemma. Then, it is possible to indicate positive constants L_1 and T_1 such that problem (1.4)–(1.6) has a unique solution in the $G_1(M) \subseteq G(M)$ domain subject to the conditions*

$$a(x) \in C^1[0, L_1], \qquad r, s \in C^2(G_1(M)).$$

Proof. We shall define the new functions p and h as follows: $p = \partial r/\partial t$ and $q = \partial s/\partial t$. The relations they satisfy are readily derived from (1.4) and (1.5) by differentiating with respect to t:

$$\begin{gathered}\frac{\partial p}{\partial x} + u(r - s)\frac{\partial p}{\partial t} = -a(x)v_1(r, s, p, q) - u_1(r, s, p, q),\\ \frac{\partial q}{\partial x} - u(r - s)\frac{\partial q}{\partial t} = a(x)v_1(r, s, p, q) + u_1(r, s, p, q), \quad p(0, t) = r_0'(t), \quad q(0, t) = s_0'(t).\end{gathered} \tag{2.1}$$

Here $v_1 = (\partial v/\partial r)p + (\partial v/\partial s)q$ and $u_1 = u'(r - s)(p - q)$.

It is easy to deduce from (1.4)–(1.6) an additional condition for p and q, which is similar to (1.6):

$$p(x, T) + q(x, T) = \frac{-2a(x)v(r(x, T), s(x, T)) - \varphi'(x)}{u(r(x, T) - s(x, T))}.$$

This relation enables one to express the function $a(x)$ via the equality

$$-a(x) = \frac{[(p(x, T) + q(x, T))u(r(x, T) - s(x, T))] + \varphi'(x)}{2v(r(x, T), s(x, T))} \tag{2.2}$$

and, hence, to exclude it from the system. It is more convenient to use symbols for subsequent writing. Let us introduce the column vectors $W(x, t) = (r(x, t),\ s(x, t),\ p(x, t),\ q(x, t))$ and $W_0(t) = (r_0(t),\ s_0(t),\ r_0'(t),\ s_0'(t))$, the diagonal matrix $D(W) = \mathrm{diag}(u(r - s),\ -u(r - s),\ u(r - s),\ -u(r - s))$, and the column vector R whose components are the right-hand sides of Eqs. (1.4) and (2.1) after $a(x)$ is replaced by relation (2.2). Finally, we come to the following relations:

$$\frac{\partial W(x, t)}{\partial x} + D(W(x, t))\frac{\partial W(x, t)}{\partial t} = R(W(x, t), W(x, T)), \quad W(0, t) = W_0(t). \tag{2.3}$$

This system does not represent a system of differential equations in the usual sense because of the quantities $W(x, T)$ on the right-hand side. Nevertheless, the system can be studied by conventional methods, because it permits the integration over the characteristics. Having made such an integration, we obtain a system of integral equations with a variable upper limit x. We shall note here only the major steps of the proof, which is similar to that given in [5, Chapter 1, § 8]. The uniqueness of the solution of system (2.3) follows from the uniqueness of the solution of a corresponding linear system for the differences in two solutions.

The fact that the solution exists can be shown using the iteration process

$$\frac{\partial W^{k+1}(x,t)}{\partial x} + D(W^k(x,t))\frac{\partial W^{k+1}(x,t)}{\partial t} = R(W^k(x,t), W^k(x,T)), \quad W^{k+1}(0,t) = W_0(t).$$

To do this, it is necessary to expand the system through differentiation of all equations with respect to x and to show, by means of the basic system, the convergence of iterations and the required smoothness of the limiting solution $W \in C^1(G_1(M))$. Hence, r, $s \in C^2(G_1(M))$, and $a \in C^1[0, L_1]$.

3. Estimation of the Conditional Stability. Let us estimate the unknown function $a(x)$ by means of the functions defined under certain *a priori* assumptions. The functions r, s, and φ are assumed to belong to the class $K(\nu, M, L, T)$ if the inequalities $(\|r\|, \|s\|, \|\varphi\|) < M$ and $\min(r+s) \geqslant \nu > 0$ hold for the norm C^2 on the intervals $[0, 2T]$ and $[0, L]$, respectively. The constant M is assumed to be universal for the entire class. We shall introduce a similar class $K_1(M_1, L)$ for the functions $a(x)$ subject to the condition $\|a\|_{C^1[0,L]} \leqslant M_1$.

Theorem 2. *Let there be two coefficients $[a^1(x)$ and $a^2(x)]$ that are the solutions of the inverse problems with given r_0^1, s_0^1, φ^1 and r_0^2, s_0^2, φ^2. If $a^1, a^2 \in K_1(M_1, L)$ and $r_0^1, s_0^1, \varphi^1, r_0^2, s_0^2, \varphi^2 \in K(\nu, M, L, T)$, the estimate*

$$\|a^1 - a^2\|_C \leqslant C(\|r_0^1 - r_0^2\|_{C^1} + \|s_0^1 - s_0^2\|_{C^1} + \|\varphi^1 - \varphi^2\|_{C^1})$$

is valid, where the constant C depends only on ν, M, M_1, L, and T.

Proof. For the coefficients $a^1(x)$ and $a^2(x)$, one can obtain the vectors W^1 and W^2 as solutions of the corresponding direct problem (2.3). We shall denote $W = W^1 - W^2$. The vector function W satisfies the following linear relations:

$$\frac{\partial W}{\partial x} + D(W^1)\frac{\partial W}{\partial t} = \langle W, e\rangle \frac{\partial W^1}{\partial t} + EW, \quad W_0(0,t) = W_0^1(t) - W_0^2(t). \tag{3.1}$$

Here $e = e(W^1, W^2)$ is the new column vector and $E = E(W^1, W^2)$ is the new matrix; and the angle brackets refer to the scalar product. The components of the vector e are elementary functions of the components of the vectors $W^1(x,t)$ and $W^2(x,t)$. The elements of the matrix E are elementary functions of the components of the vectors $W^1(x,t)$, $W^2(x,t)$, $W^1(x,T)$, and $W^2(x,T)$. Because of the awkwardness, we do not give the exact representation; for subsequent analysis, only the fact that the vector $e(x,t)$ and the matrix $E(x,t)$ belong to $C^1(G)$ is important. Integrating over the characteristics, we pass from (3.1) to the system of integral equations

$$W(x,t) = W_0(t_1(x,t)) + \int_0^x \Big[\langle W, e\rangle \frac{\partial W^1}{\partial t} + EW\Big](\xi_i, x, t)\, d\xi_i \tag{3.2}$$

(the subscript i refers to the corresponding characteristic). We shall introduce a norm for the function $f(x,t)$:

$$\|f\|(x) = \max_{0<t<2T}(|f(x,t)|, |\partial f(x,t)/\partial t|),$$

which is the standard norm for the vector

$$\|W\|(x) = \sum_i \|(W)_i\|(x).$$

Let C_i be different constants. After conventional procedures (see [4]), Eq. (3.2) yields the estimate $\|W\|(x) \leqslant C_1\|W_0\|_{C^1}$. Then, from (2.2) it is not difficult to obtain the inequality

$$|a^1(x) - a^2(x)| \leqslant C_2\|W\|(x) + C_3\|\varphi^1 - \varphi^2\|_{C^1},$$

which yields the desired estimate.

4. Some Generalizations. In formulating the problem in Sec. 2, some assumptions were made to facilitate mathematical procedures. Let us consider the more general cases of problem formulation.

The width of the free surface can be a function of x $[B = B(x)]$. This leads to the appearance of a new term on the right-hand side of (1.4):

$$\frac{\partial r}{\partial x} + u(r-s)\frac{\partial r}{\partial t} = -a(x)v(r,s) + \frac{\partial B}{\partial x}v_2(r,s), \quad \frac{\partial s}{\partial x} - u(r-s)\frac{\partial s}{\partial t} = a(x)v(r,s) + \frac{\partial B}{\partial x}v_2(r,s).$$

If the function $B(x)$ is sufficiently smooth [for example, $B(x) \in C_2[0, L])$], the proofs of Theorems 1 and 2 remain unchanged.

One can study the flow behavior not only in a rectangular channel. For example, let the channel be of an arbitrary shape, $B = B(h, x)$. We denote the level of the free water surface by $z(x, t)$ and the level of the bottom at the point x by $z_b(x)$ and introduce a parameter of the channel cross-section area:

$$\omega(z, x) = \int_0^{h(z,x)} B(\xi, x)\, d\xi, \quad h(z, x) = z - z_b(x).$$

The Riemann invariants should then be introduced by the following formulas:

$$r = \int_0^z (gB(\xi, x)\omega(\xi, x))^{1/2}\, d\xi + Q(x, t), \quad s = -\int_0^z (gB(\xi, x)\omega(\xi, x))^{1/2}\, d\xi + Q(x, t).$$

Instead of Eq. (1.4), we obtain

$$\frac{\partial r}{\partial x} + \left(\frac{g\omega}{B}\right)^{1/2} \frac{\partial r}{\partial t} = -a(x)v(r, s) + g\omega \int_0^z \frac{\partial}{\partial x}(gB\omega)^{1/2}\, d\xi,$$

$$\frac{\partial s}{\partial x} - \left(\frac{g\omega}{B}\right)^{1/2} \frac{\partial s}{\partial t} = a(x)v(r, s) + g\omega \int_0^z \frac{\partial}{\partial x}(gB\omega)^{1/2}\, d\xi.$$

In this case, the additional information also has the more complicated form

$$r(x, T) - s(x, T) = 2\int_0^{\varphi_0} (gB(\xi, x)\omega(\xi, x))^{1/2}\, d\xi = \varphi(x).$$

Nevertheless, all the above results hold for this formulation of the problem as well.

The following remark is concerned with the additional information (1.3). The water-surface level can be measured not only on the straight lines $t = T$, but also on the more general straight lines in the plane x, t. Let there be an observer who moves with a given constant velocity v_0 along the channel from the point x_0. It is assumed that v_0 can differ from the flow velocity. Additional information should then be specified as a function $h(x, (x - x_0)/v_0) = \varphi_0(x)$.

The author are grateful to O. F. Vasil'ev and A. F. Voyevodin for the formulation of the problem and helpful discussions.

REFERENCES

1. P. P. Chugaev, *Hydraulics* [in Russian], Energoizdat, Leningrad (1982).
2. F. M. Henderson, *Open Channel Flow*, Collier-Macmillan, London (1966).
3. A. F. Voyevodin and V. S. Nikiforovskaya, "Numerical method of solving some inverse problems of hydraulics," in: *Water Resources* [in Russian], **3** (1981), pp. 114–118.
4. V. G. Romanov, *Inverse Problems of Mathematical Physics* [in Russian], Nauka, Moscow (1984).
5. B. L. Rozhdestvenskii and N. N. Yanenko. *Systems of Quasi-Linear Equations and Their Applications to Gas Dynamics* [in Russian], Nauka, Moscow (1978).